MATH 320 Unit 5 Exam Name:

Instructions:

Please write your answers on separate paper. Please write clearly and legibly, using a large font and plenty of white space (I need room to put my comments). Staple all your pages together, with your problems in order, when you turn in your exam. Make clear what work goes with which problem. Put your name on every page. To get credit, you must show adequate work to justify your answers. If unsure, show the work. No outside materials are permitted on this exam – no notes, papers, books, calculators, phones, smartwatches, or computers – only pens and pencils. You may freely use the contents of the box on the reverse side, but not any other results we may have proved. Each problem is out of 10 points, 40 points maximum. You have 30 minutes.

1. Let R, S be rings. Consider the function $f : R \times S \to R$ given by $f((a, b)) = a$. Prove that f is a homomorphism.
2. Consider the function $f : \mathbb{C} \to \mathbb{C}$ given by $f(a + bi) = b + ai$. Prove or disprove that f is an isomorphism.
3. Let R be a ring, and let I, J be ideals. Prove that $I \cap J$ is an ideal.
4. Prove or disprove that $\mathbb{Z}_8$ and $\mathbb{Z}_2 \times \mathbb{Z}_4$ are isomorphic.

Let R, S be rings. The Cartesian product $R \times S$ forms a ring via $(r,s) \oplus (r',s') = (r+r', s+s')$ and $(r,s) \odot (r',s') = (rr', ss')$, where $r + r'$ and rr' are using the operations in R and $s + s', ss'$ are using the operations in S.

Let R be a ring and $S \subseteq R$. We call S a *subring* of R if it is closed under addition and multiplication, contains 0_R, and for every $a \in S$ the solution of $a + x = 0_R$ is in S (not just in R).

Let R, S be rings, and $f : R \to S$ a function. We call f a *homomorphism* if it satisfies

$$\text{For all } a, b \in R,\ f(a+b) = f(a) + f(b) \text{ and } f(ab) = f(a)f(b).$$

If a homomorphism is also a bijection (i.e. is surjective and injective), we call it a *isomorphism*, and say that the rings are *isomorphic*.

Basic Homomorphism Properties Theorem: Let $f : R \to S$ be a homomorphism. Then (i) $f(0_R) = 0_S$; and (ii) For all $a \in R$, $f(-a) = -f(a)$; and (iii) For all $a, b \in R$, $f(a - b) = f(a) - f(b)$.

Surjective Homomorphism Properties Theorem: Let $f : R \to S$ be a surjective homomorphism, and suppose 1_R exists. Then (i) $f(1_R) = 1_S$; and (ii) If $u \in R$ is a unit (in R), then $f(u)$ is a unit in S and $f(u)^{-1} = f(u^{-1})$.

Ring Image Theorem: Let $f : R \to S$ be a homomorphism. Then $Im(f) = \{f(r) : r \in R\}$ is a subring of S.

Let I be a subring of R. We call I an *ideal* if it also satisfies

$$\text{For all } r \in R, a \in I,\ \text{then } ra \in I \text{ and } ar \in I.$$

Let R be a commutative ring with identity, let $c \in R$. We define $\{rc : r \in R\}$, writing (c). We call this subset of R the *principal ideal generated by c*.

Let R be a ring with ideal I, and let $a, b \in R$. We say that *a is congruent to b modulo I*, writing $a \equiv b \pmod{I}$, if $a - b \in I$.

Let R be a ring with ideal I, and let $a \in R$. The *congruence class (or equivalence class, or coset) of a modulo I*, written $a + I$, is the set $\{b \in R : b \equiv a \pmod{I}\}$. We define R/I to be the set of equivalence classes modulo I.